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Examples of Chebyshev Sets in Matrix Spaces

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Let A be a matrix in $\mathbb{C}^{n\times n}$ and let $U\Sigma V^*$ be its singular value decomposition. The authors prove that for each $1\leqslant k\leqslant n$ the set $\mathscr{S}_1^{(k)}=\{S\in\mathbb{C}^{n\times n}: \sum_{1\leqslant j_1<\cdots<j_k\leqslant n}\sigma_{j_1}(S)\,\sigma_{j_2}(S)\cdots\sigma_{j_k}(S)\leqslant 1\}$ is a Chebyshev set in $\mathbb{C}^{n\times n}$ endowed with the spectral norm and that the metric projection is globally Lipschitz-continuous.

In the 1980s the senior author studied various problems on matrix approximation; he was, in particular, interested in Chebyshev sets and suns. Approximation in matrix spaces proved to be rich in providing simple, but impressive examples as the papers [1, 4, 5, 7] show. We want to take up the results in [3, 4], extend, and re-prove them.

We denote by $\mathbb{C}^{n\times m}$ the vector space of complex $n\times m$ matrices over $\mathbb{C}$, $n, m\in\mathbb{N}$, with elements $A, B, \ldots$. For $A\in\mathbb{C}^{n\times p}$ and $B\in\mathbb{C}^{p\times m}$, $n, m, p\in\mathbb{N}$, A^* denotes the adjoint of A in $\mathbb{C}^{p\times n}$ and AB the matrix product of A and B in $\mathbb{C}^{n\times m}$. Instead of $\mathbb{C}^{n\times 1}$ we write $\mathbb{C}^n$ the vector space of complex column vectors; we also write $z, u, \ldots$ to denote its elements.

By l_n^2 we denote $\mathbb{C}^n$ endowed with the Euclidian norm $|\cdot|_2$, and by $\mathscr{L}(l_n^2)$ the vector space of linear transformations of l_n^2 into itself. $\mathscr{L}(l_n^2)$ can be identified with $\mathbb{C}^{n\times n}$ endowed with the l^2-operator norm, where $A\in\mathbb{C}^{n\times n}$ acts on $z\in\mathbb{C}^n$ via the matrix product Az. Since it will not lead to

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any ambiguity, in the following we shall write $\mathbb{C}^n$ and $\mathbb{C}^{n\times n}$ instead of l_n^2 and $\mathscr{L}(l_n^2)$, respectively.

For an $A \in \mathbb{C}^{n\times n}$, we denote its *singular value decomposition* (SVD) by $U\Sigma V^*$, U and V are unitary matrices in $\mathbb{C}^{n\times n}$, and $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, ..., \sigma_n)$, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$. Σ is uniquely determined by A; its elements are the so-called *singular values* of A; to indicate the dependence on A, we will write $\Sigma(A)$ and $\sigma_1(A), \sigma_2(A), ..., \sigma_n(A)$, respectively. If A is non-singular, then UV^* is uniquely defined.

Here we will be interested in best approximation on $\mathbb{C}^{n\times n}$ w.r.t. the spectral norm. It will, however, be appropriate to consider more generally *unitarily invariant norms* $\|\cdot\|$; in particular, we will consider the *Schatten p-norms*, where

$$\text{for all} \quad A \in \mathbb{C}^{n\times n}, \qquad \|A\|_p^p = \|\Sigma\|_p^p = \sum_{j=1}^{n} \sigma_j^p.$$

For $p=1$ we speak of the (classical) Schatten norm $\|\cdot\|_1$, while for $p=\infty$, $\|\cdot\|_\infty$ just denotes the spectral norm on $\mathbb{C}^{n\times n}$. Introducing on $\mathbb{C}^{n\times n}$ the *sesqui-linear form*

$$\text{for all} \qquad A, B \in \mathbb{C}^{n\times n}, \qquad \langle A, B\rangle = \operatorname{trace} A^*B,$$

$\mathbb{C}^{n\times n}$ becomes an inner product space isomorphic to $\mathbb{C}^{n^2}$. The associated norm $\|\cdot\|_2$, the so-called Frobenius norm, is nothing but the 2-norm

$$\text{for all} \quad A \in \mathbb{C}^{n\times n}, \qquad \sum_{j,k=1}^{n} |a_{jk}|^2 = \|A\|_2^2 = \|\Sigma\|_2^2 = \sum_{j=1}^{n} \sigma_j^2.$$

For $k = 1, 2, ..., n$, we define

$$\mathscr{S}_1^{(k)} = \left\{ S \in \mathbb{C}^{n\times n} : \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n} \sigma_{j_1}(S)\, \sigma_{j_2}(S) \cdots \sigma_{j_k}(S) \leqslant 1 \right\}.$$

If $k=1$, $\mathscr{S}_1^{(1)}$ coincides with the l_1-unit ball in $\mathbb{C}^{n\times n}$; i.e., $\mathscr{S}_1^{(1)} = \{S \in \mathbb{C}^{n\times n} : \|S\|_1 \leqslant 1\}$, while for $k=n$ $\mathscr{S}_1^{(n)}$ $\{S \in \mathbb{C}^{n\times n} : |\det S| \leqslant 1\}$. It was proved in [4], that $\mathscr{S}_1^{(n)}$ is a Chebyshev set in $\mathbb{C}^{n\times n}$ w.r.t. $\|\cdot\|_\infty$ and that the metric projection $P^\infty_{\mathscr{S}_1^{(n)}}$ is globally Lipschitz-continuous. We will re-prove the result and extend it; our proof is different and more elementary.

THEOREM. *For $1 \leqslant k \leqslant n$, the set $\mathscr{S}_1^{(k)}$ is a Chebsyshev set in $\mathbb{C}^{n\times n}$ w.r.t. the spectral norm.*

More precisely, if $A \in \mathbb{C}^{n \times n} \backslash \mathscr{S}_1^{(k)}$ *and if* $U\Sigma V^*$ *is a SVD of* A, *then the distance of* A *from* $\mathscr{S}_1^{(k)}$ *is given by*

$$d^{(k)} = \min \left\{ t > 0 : \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n} (\sigma_{j_1} - t)_+ (\sigma_{j_2} - t)_+ \cdots (\sigma_{j_k} - t)_+ = 1 \right\}.$$

With $r \in \mathbb{N}, k \leqslant r \leqslant n$, *such that* $\sigma_{r+1} < d^{(k)} \leqslant \sigma_r (\sigma_{n+1} = 0)$, *and*

$$\tilde{\Sigma}^{(k)} = \operatorname{diag}[\sigma_1 - d^{(k)}, ..., \sigma_r - d^{(k)}, 0, ..., 0],$$

$\tilde{A}^{(k)} = U\tilde{\Sigma}^{(k)}V^*$ *is the unique element of best approximation of* A *in* $\mathscr{S}_1^{(k)}$.

Moreover, the metric projection onto $\mathscr{S}_1^{(k)}$ *is globally Lipschitz-continuous.*

Before we get into proving the theorem, let us remark that the element of best approximation (briefly, el. b. appr.) is well-defined. It follows from the result stated below; although it is well known, we consider it worthwile to point this out.

Let $A \in \mathbb{C}^{n \times n}$, $A \neq 0$, *and let* $U\Sigma V^*$ *and* $X\Sigma Y^*$ *be two SVDs of* A, *where*

$$\Sigma = \operatorname{diag}(\underbrace{\sigma_1, ..., \sigma_1}_{n_1}, \underbrace{\sigma_2, ..., \sigma_2}_{n_2}, ..., \underbrace{\sigma_r, ..., \sigma_r}_{n_r}, 0, ..., 0),$$

$n_1 + \cdots + n_r = \operatorname{rank}(A)$. *Then there exist unitary matrices* P *and* Q *in* $\mathbb{C}^{n \times n}$, *such that* $X = UP$ *and* $Y = VQ$, *and*

$$P = \operatorname{diag}(W_1, ..., W_r, P_0) \qquad \text{and} \qquad Q = \operatorname{diag}(W_1, ..., W_r, Q_0),$$

where P_0, Q_0, *and* $W_j, 1 \leqslant j \leqslant r$, *are unitary matrices in* $\mathbb{C}^{n_0 \times n_0}$, $n_0 + \operatorname{rank}(A) = n$, *and in* $\mathbb{C}^{n_j \times n_j}$, *respectively. If* A *is nonsingular then* $UV^* = XY^*$.

To prove the theorem it suffices to consider Σ instead of A and to prove that $\tilde{\Sigma}^{(k)}$ is the el. b. appr. of Σ in $\mathscr{S}_1^{(k)}$. Concerning the proof itself, we will prove more strongly that the *strict Kolmogorov condition*, namely,

$$\text{for all} \quad S \in \mathscr{S}_1^{(k)}, S \neq \tilde{\Sigma}^{(k)}, \qquad \min_{\substack{|q|_2 = 1; \\ q \in \mathbb{C}^r \times \{0\}^{n-r}}} \Re q^* (S - \tilde{\Sigma}^{(k)})\, q < 0 \tag{1}$$

holds, which implies uniqueness.

The following remark is in order. Let X be real or complex, normed vector space with norm $|\cdot|_X$, and let K be a (closed) subset of X. We denote the metric projection of X onto K by P_K. For $x \in X \backslash K$ and $k \in P_K(x)$, k is said to be a *solar point* of x in K, if $k \in P_K(x_t)$ for each point x_t on the ray from k through x; clearly, $x_t = k + t(x - k)$, $t \in \mathbb{R}_+$. If for some subset K of X each $x \in X \backslash K$ has a solar point, then K is said to be a *sun* in X in the sense of L. P. Vlasov. Solar points are best described by use of the semi-inner product $\langle \cdot, \cdot \rangle_s$ defined on $X \times X$;

for $x, y \in X$,

$$\langle y, x \rangle_s = \lim_{t \to 0+} \frac{|x+ty|_X^2 - |x|_X^2}{2t} = \max\{ \Re\langle w, y \rangle : w \in \Phi(x)\},$$

Φ is the *duality map* on X; i.e., for each $x \in X : \Phi(x) = \{w \in X^* : \Re\langle w, x \rangle = |x|_X^2 = |w|_{X^*}^2\}$. In the definition we consider X to be a vector space over the reals; this is often done in approximation theory. To distinguish the complex space X from the space X considered as a real vector space, one usually writes X_r—the subscript, however, will be dropped, when there will be no confusion. If X is an inner product space, the semi-inner products reduces to the inner product. The semi-inner product has many properties similar to the inner product, but it is in general far more restrictive, see, e.g., [2] for details.

To get back to the characterization of solar points, under the conditions given above, a point $k \in K$ is a solar point for $x \in X \backslash K$ if and only if

$$\text{for all} \quad k' \in K, \qquad 0 \leqslant \langle k - k', x - k \rangle_s; \tag{2}$$

the condition can be, and will be, interpreted as a *Kolmogrov condition*. All this is well-known and well documented.

To conclude the remark, in the finite dimensional setting a Chebyshev set is a Chebyshev sun, and the metric projection is continuous.

The proof of the theorem is based upon the following lemmas.

LEMMA 1. *Let* $\Sigma = \operatorname{diag}(\sigma_1, ..., \sigma_n)$, $\sigma_1 = \cdots = \sigma_r > \sigma_{r+1} \geqslant \cdots \geqslant \sigma_n \geqslant 0$. *Then*

$$\Phi_\infty(\Sigma) = \operatorname{co}\{qq^* \in \mathbb{C}^{r \times r} : q \in \mathbb{C}^r \times \{0\}^{n-r} \text{ and } |q|_2^2 = \sigma_1\},$$

"co" *means the (closed) convex hull, and consequently,*

$$\text{for all} \quad B \in \mathbb{C}^{n \times n}, \qquad \langle B, \Sigma \rangle_\infty = \max_{\substack{|q|_2^2 = \sigma_1 \\ q \in \mathbb{C}^r \times \{0\}_{n-r}}} \Re q^* B q.$$

If $A \in \mathbb{C}^{n \times n}$ *has* $U\Sigma V^*$ *to be its SVD, then* $\Phi_\infty(A) = U\Phi_\infty(\Sigma)V^*$.

For a proof see [3]. Note that we used the subscript ∞ to indicate that we consider the spectral norm on $\mathbb{C}^{n \times n}$.

Our second lemma is concerned with unitarily invariant norms in general.

LEMMA 2. *For any unitarily invariant norm* $\|\cdot\|$ *on* $\mathbb{C}^{n\times n}$

$$\text{for all} \quad A, B \in \mathbb{C}^{n\times n}, \qquad \|\Sigma(A)-\Sigma(B)\| \leqslant \|A-B\|.$$

A proof of the basic inequality is, e.g. given in the monograph of R. A. Horn and Ch. R. Johnson [6, Theorem 7.4.51]. For the Frobenius norm the inequality is known as the inequality of Hoffmann and Wielandt.

Proof of the Theorem. For the proof we drop the superscript (k); this will not lead to any confusion.

We shall first prove uniqueness. Under the assumption of the theorem we have that

$$\Sigma-\tilde{\Sigma}=\operatorname{diag}[d, ..., d, \sigma_{r+1}, ..., \sigma_n], \qquad \sigma_{r+1}<d\leqslant\sigma_r,$$

where d denotes the distance of Σ from $\mathcal{S}_1$. By Lemma 1, Kolmogorov's condition (2) reads

for all $S\in\mathcal{S}_1$, $S\neq\tilde{\Sigma}$,

$$0<\langle\tilde{\Sigma}-S, \Sigma-\tilde{\Sigma}\rangle_\infty = d\cdot \max_{\substack{q\in\mathbb{C}^r\times\{0\}^{n-r}\\ |q|_2=1}} \Re q^*(\tilde{\Sigma}-S)\,q. \tag{3}$$

We will use Lemma 2 to conclude that

$$\langle\tilde{\Sigma}-\Sigma(S), \Sigma-\tilde{\Sigma}\rangle_\infty \leqslant \langle\tilde{\Sigma}-S, \Sigma-\tilde{\Sigma}\rangle_\infty. \tag{4}$$

Indeed, $\|\Sigma-\tilde{\Sigma}+t(\tilde{\Sigma}-\Sigma(S))\|_\infty \leqslant \|\Sigma-\tilde{\Sigma}+t(\tilde{\Sigma}-S)\|_\infty$ for all $t\in\mathbb{R}_+$, and the inequality follows.

Let us assume that $S\in\mathcal{S}_1$ is such that $\Sigma(S)\neq\tilde{\Sigma}$. We claim that

$$0<\max_{\substack{q\in\mathbb{C}^r\times\{0\}_{n-r}\\ |q|_2=1}} \Re q^*(\tilde{\Sigma}-\Sigma(S))\,q. \tag{5}$$

Since

$$\sum_{1\leqslant j_1<\cdots<j_k\leqslant n} \sigma_{j_1}(S)\cdots\sigma_{j_k}(S)\leqslant 1 \qquad \text{and} \qquad \sum_{1\leqslant j_1<\cdots<j_k\leqslant n} \tilde{\sigma}_{j_1}\cdots\tilde{\sigma}_{j_k}=1,$$

there is at least one index in $\{1, ..., r\}$, say, j_0 for which $\sigma_{j_0}(S) < \tilde{\sigma}_{j_0}$. For otherwise, for all j $\tilde{\sigma}_j \leqslant \sigma_j(S)$, but then

$$1 = \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \tilde{\sigma}_{j_1} \cdots \tilde{\sigma}_{j_k} \leqslant \sum_{1 \leqslant j_1 < \cdots < j_k \leqslant n} \sigma_{j_1}(S) \cdots \sigma_{j_k}(S) \leqslant 1,$$

implies that for all j $\sigma_j(S) = \tilde{\sigma}_j$, contradicting the condition $\Sigma(S) \neq \tilde{\Sigma}$. Hence $\sigma_{j_0}(S) < \tilde{\sigma}_{j_0}$ and for $q = e_{j_0}$, the j_0's natural basis element in $\mathbb{C}^r$, $e_{j_0}^*(\tilde{\Sigma} - \Sigma(S))\, e_{j_0} = \tilde{\sigma}_{j_0} - \sigma_{j_0}(S) > 0$, which proves (5). From (5) and (4) we obtain (3).

Next assume that $S \in \mathscr{S}_1$ is such that $S \neq \tilde{\Sigma}$, but $\Sigma(S) = \tilde{\Sigma}$. Assume Kolmogorov's condition (3) does not hold; i.e.,

$$\text{for all} \quad q \in \mathbb{C}^r \times \{0\}^{n-r}, \qquad \Re q^* (\tilde{\Sigma} - S)\, q \leqslant 0. \tag{6}$$

Setting $S = B + iC$ with B and C Hermitian $(B = (S + S^*)/2$ and $C = -i(S - S^*)/2)$, then (6) just reads

$$\text{for all} \quad q \in \mathbb{C}^r, \qquad q^*(\tilde{\Sigma}_r - B_r)\, q \leqslant 0 \quad \text{or} \quad q^*(B_r - \tilde{\Sigma}_r)\, q \geqslant 0;$$

i.e., the matrix $B_r - \tilde{\Sigma}_r$ is positive semi-definite, where the subscript r indicates that we restrict the matrix to the rth principal submatrix. This forces $\tilde{\sigma}_j \leqslant b_{jj}$ for $1 \leqslant j \leqslant r$, and consequently,

$$\|\tilde{\Sigma}\|_2^2 = \sum_{j=1}^{r} \tilde{\sigma}_j^2 \leqslant \sum_{j=1}^{r} \tilde{\sigma}_j^2 + \sum_{j=r+1}^{n} b_{jj}^2 + \sum_{j \neq k} |b_{jk}|^2 \leqslant \|B\|_2^2 \leqslant \|S\|_2^2 = \|\tilde{\Sigma}\|_2^2.$$

Hence, all b_{jj} vanish for $r + 1 \leqslant j \leqslant n$ as well as all $b_{jk}, j \neq k$; i.e., $B = \tilde{\Sigma}$.

Note that $\Re \langle \Sigma, iC \rangle = 0$, since C is Hermitian. It follows that $\|S\|_2^2 = \|\tilde{\Sigma} + iC\|_2^2 = \|\tilde{\Sigma}\|_2^2 + \|C\|_2^2 = \|\Sigma(S)\|_2^2 = \|\tilde{\Sigma}\|_2^2$, hence $C = 0$, and consequently $S = \tilde{\Sigma}$ a contradiction.

Thus, the strict Kolmogorov condition (1) is satisfied in both cases, proving the first part of the theorem. It remains to prove that the metric projection is globally Lipschitz-continuous. As we remarked above, it is continuous, but we have more.

Let us at first consider diagonal matrices only, say, Σ and Σ' not in $\mathscr{S}_1$ with el. b. appr. $\tilde{\Sigma}$ and $\tilde{\Sigma}'$ and distances d and d', respectively. Note that for any $1 \leqslant j \leqslant n$

$$|\tilde{\sigma}_j - \tilde{\sigma}'_j| = |(\sigma_j - d)_+ - (\sigma'_j - d')_+| \leqslant |\sigma_j - \sigma'_j| + |d - d'|,$$

and consequently,

$$\|\tilde{\Sigma}-\tilde{\Sigma}'\|_2 \leqslant \|\Sigma-\Sigma'\|_2+\sqrt{n}\,|d-d'|,$$

but $|d-d'| \leqslant \|\Sigma-\Sigma'\|_\infty \leqslant \|\Sigma-\Sigma'\|_2$, giving

$$\|\tilde{\Sigma}-\tilde{\Sigma}'\|_2 \leqslant 2\sqrt{n}\,\|\Sigma-\Sigma'\|_2. \tag{7}$$

To consider the general situation, let A, A' be two matrices in $\mathbb{C}^{n\times n}\setminus\mathcal{S}_1$. W.l.o.g., we may assume that $A=\Sigma$, Σ as given above, and that $A'=XE'Y^*$. Then the metric projection of Σ and A' are $\tilde{\Sigma}$ and $X\tilde{\Sigma}'Y^*$, respectively. We have to estimate the difference

$$\|\tilde{\Sigma}-X\tilde{\Sigma}'Y^*\|$$

w.r.t. some (unitarily invariant) norm, say the 2-norm again. By the triangular inequality

$$\|\tilde{\Sigma}-X\tilde{\Sigma}'Y^*\|_2 \leqslant \|\tilde{\Sigma}-X\tilde{\Sigma}Y^*\|_2+\|X(\tilde{\Sigma}-\tilde{\Sigma}')\,Y^*\|_2.$$

For the second term of the right-hand side we have

$$\|X(\tilde{\Sigma}-\tilde{\Sigma}')\,Y^*\|_2=\|\tilde{\Sigma}-\tilde{\Sigma}'\|_2 \leqslant 2\sqrt{n}\,\|\Sigma-\Sigma'\|_2 \leqslant 2\sqrt{n}\,\|\Sigma-A'\|_2.$$

The two estimates follow from (7) and Lemma 2, respectively.

To estimate the first term, we take advantage of an estimate obtained in [3].

Let Σ and $\tilde{\Sigma}$ and d be given as above, and let $\Sigma+E=X\Sigma Y^*$ be a perturbation of Σ so that its SVD leaves the singular values unchanged. Then

$$\|\tilde{\Sigma}-X\tilde{\Sigma}Y^*\|_2 \leqslant \|E\|_2.$$

The estimate holds for any $\tilde{\Sigma}=\operatorname{diag}(\tilde{\sigma}_1, ..., \tilde{\sigma}_n)$, $\tilde{\sigma}_1 \geqslant \cdots \geqslant \tilde{\sigma}_n \geqslant 0$, for which for $1\leqslant j\leqslant l\leqslant n$ the inequality $\tilde{\sigma}_j\pm\tilde{\sigma}_l\leqslant\sigma_j\pm\sigma_l$ holds. In our case, $\tilde{\sigma}_j=\sigma_j-d$ for $1\leqslant j\leqslant r$ and $\tilde{\sigma}_j=0$ for $r+1\leqslant j\leqslant n$, where $\sigma_{r+1}<d\leqslant\sigma_r$.

As a consequence, we obtain

$$\begin{aligned}\|\tilde{\Sigma}-X\tilde{\Sigma}Y^*\|_2 &\leqslant \|\Sigma-X\Sigma Y^*\|_2 \leqslant \|\Sigma-X\Sigma'Y^*\|_2+\|X(\Sigma'-\Sigma)\,Y^*\|_2\\ &\leqslant 2\,\|\Sigma-A'\|_2.\end{aligned}$$

Adding the two estimates, proves Lipschitz-continuity and completes the proof of the theorem.

Instead of the class $\mathscr{S}_1^{(k)}$, $1 \leqslant k \leqslant n$, let us consider, more generally, the class

for $\delta > 0$,

$$\mathscr{S}_\delta^{(k)} = \left\{ S \in \mathbb{C}^{n \times n} : \sum_{1 \leqslant j_1 < j_2 < \cdots < j_k \leqslant n} \sigma_{j_1}(S)\, \sigma_{j_2}(S) \cdots \sigma_{j_k}(S) \leqslant \delta \right\}.$$

It follows from above that $\mathscr{S}_\delta^{(k)}$ is a Chebyshev set in $\mathbb{C}^{n \times n}$ w.r.t. the spectral norm for each $\delta > 0$. Moreover, repeating the proof of Lipschitz-continuity of the metric projection, we see that it remains unchanged when we replace $P^\infty_{\mathscr{S}_1^{(k)}}$ by $P^\infty_{\mathscr{S}_\delta^{(k)}}$; i.e., $P^\infty_{\mathscr{S}_1^{(k)}} : \mathbb{C}^{n \times n} \to \mathscr{S}_\delta^{(k)}$ is globally Lipschitz-continuous with Lipschitz constant independent of $1 \leqslant k \leqslant n$ and $\delta > 0$.

If we allow δ to converge towards zero, the class $\mathscr{S}_\delta^{(k)}$ converges to

$$\mathscr{S}_0^{(k)} = \{ S \in \mathbb{C}^{n \times n} : \operatorname{rank} S \leqslant k - 1 \}.$$

Trivially, $\mathscr{S}_0^{(1)}$ just reduces to $\{0\}$, while $\mathscr{S}_0^{(n)}$ is the class of singular matrices.

Let $A \in \mathbb{C}^{n \times n} \backslash \mathscr{S}_0^{(k)}$, and let $U \Sigma V^*$ be its SVD. Then, its singular value $\sigma_k \neq 0$ and $A \notin \mathscr{S}_\delta^{(k)}$ for $\delta > 0$ sufficiently small. By the theorem the distance $\operatorname{dist}_\infty(A; \mathscr{S}_\delta^{(k)})$ is given by

$$d_\delta^{(k)} = \min \left\{ t > 0 : \sum_{1 \leqslant j_1 \leqslant \cdots \leqslant j_k \leqslant n} (\sigma_{j_1} - t)_+ \cdots (\sigma_{j_k} - t)_+ = \delta \right\},$$

and $U \tilde{\Sigma}_\delta^{(k)} V^*$ is the unique el. b. appr. of A from $\mathscr{S}_\delta^{(k)}$ with

$$\tilde{\Sigma}_\delta^{(k)} = \operatorname{diag}[\sigma_1 - d_\delta^{(k)}, ..., \sigma_k - d_\delta^{(k)}, 0, ..., 0].$$

Since $d_\delta^{(k)}$ converges to σ_k as $\delta \to 0+$, $\operatorname{dist}(A; \mathscr{S}_0^{(k)}) = \sigma_k$ and $U \tilde{\Sigma}_0^{(k)} V^*$ is an el. b. appr. of A in $\mathscr{S}_0^{(k)}$, where $\tilde{\Sigma}_0 = \operatorname{diag}[\sigma_1 - \sigma_k, ..., \sigma_{k-1} - \sigma_k, 0, ..., 0]$. More precisely,

Corollary. *For $1 \leqslant k \leqslant n$ $\mathscr{S}_0^{(k)}$ is a sun in $\mathbb{C}^{n \times n}$ w.r.t. the spectral norm. If $A \in \mathbb{C}^{n \times n} \backslash \mathscr{S}_0^{(k)}$ and if $U \Sigma V^*$ is its SVD, then*

$$\operatorname{dist}_\infty(A; \mathscr{S}_0^{(k)}) = \sigma_k \qquad \text{and} \qquad U \tilde{\Sigma}_0^{(k)} V^* \text{ is a solar point of } A \text{ in } \mathscr{S}_0^{(k)},$$

$\tilde{\Sigma}_0^{(k)}$ *being* $\operatorname{diag}[\sigma_1 - \sigma_k, ..., \sigma_{k-1} - \sigma_k, 0, ..., 0]$.

In addition, $\mathbb{C}^{n \times n} \ni A = U \Sigma V^* \mapsto U \tilde{\Sigma}_0^{(k)} V^*$ is a Lipschitz-continuous selection of $P^\infty_{\mathscr{S}_0^{(k)}}$.

This re-proves results in [3, 4]. In this context we would like to add the following observation.

As we remarked above, $\mathscr{S}_1^{(1)}$ is just equal to the l_1-unit ball $\bar{b}_1^{(1)}(0)$ of $\mathbb{C}^{n\times n}$. $\mathscr{S}_1^{(1)}$ being a Chebyshev set in $\mathbb{C}^{n\times n}$ w.r.t. $\|\cdot\|_\infty$ then means that for each $A\in\mathbb{C}^{n\times n}\backslash\mathscr{S}_1^{(1)}$ the *distance ball* $b_{\mathscr{S}_1^{(1)}}^{(\infty)}(A)$ of A touches $\bar{b}_1^{(1)}(0)$ exactly at $P_{\mathscr{S}_1^{(1)}}(A) - b_{\mathscr{S}_1^{(1)}}^{(\infty)}(A)$ denotes the closed ball centered at A with radius $\operatorname{dist}_\infty(A; \mathscr{S}_1^{(1)})$. Interpreting this statement right, leads to the following.

COROLLARY. *The l_∞-unit ball $\bar{b}_1^{(\infty)}(0)=\{S\in\mathbb{C}^{n\times n}:\sigma_1(S)\leqslant 1\}$ is a Chebyshev set in $\mathbb{C}^{n\times n}$ w.r.t. the Schatten 1-norm. Moreover, the metric projection is globally Lipschitz-continuous.*

We conclude our investigation with the following remarks. Let $A\in\mathbb{C}^{n\times n}$ be Hermitian; i.e., $A=U\Lambda U^*$, $\Lambda=\operatorname{diag}[\lambda_1,\lambda_2,...,\lambda_n]$ and U unitary, where the λ_j's are the eigenvalues of A counting their multiplicities and ordered such that $|\lambda_1|\geqslant|\lambda_2|\geqslant\cdots\geqslant|\lambda_n|\geqslant 0$. The selected el.'s b. appr. of A in the classes $\mathscr{S}_1^{(k)}$ and $\mathscr{S}_0^{(k)}$, $1\leqslant k\leqslant n$, are then given by

$$\tilde{A}^{(k)}=U\tilde{\Lambda}^{(k)}U^* \qquad \text{and} \qquad \tilde{A}_0^{(k)}=U\tilde{\Lambda}_0^{(k)}U^*, \text{ respectively.}$$

Indeed, if we set $D_\Lambda=\operatorname{diag}[\operatorname{sign}\lambda_1,\operatorname{sign}\lambda_2,...,\operatorname{sign}\lambda_n]$ it follows that $UD_\Lambda\Sigma_A U^*$ is the SVD of A with $\Sigma_A=D_\Lambda\Lambda$, and consequently, $\tilde{\Lambda}^{(k)}=D_\Lambda\tilde{\Sigma}^{(k)}$ and $\tilde{\Lambda}_0^{(k)}=D_\Lambda\bar{\Sigma}_0^{(k)}$, respectively, which proves our claim. A similar statement holds true if we assume $A\in\mathbb{C}^{n\times n}$ to be (complex) symmetric, but let us stick to the Hermitian case.

As we pointed out in our general discussion an approximation in a complex normed vector space X, approximation in X means approximation in X_r. For $\mathbb{C}_r^{n\times n}$ the subset of Hermitian matrices then forms a linear subspace, say, $\mathbb{H}^{n\times n}$. It follows from above that *rank approximation* in $\mathbb{H}^{n\times n}$, endowed with the spectral norm, means first rank approximation in $\mathbb{C}^{n\times n}$ and then restricting the results to $\mathbb{H}^{n\times n}$.

On the other hand, instead of dealing with the class $\mathscr{S}_1^{(k)}\cap\mathbb{H}^{n\times n}$, $1\leqslant k\leqslant n$, it seemed to us more natural to replace it by

$$\mathscr{H}_1^{(k)}=\left\{S\in\mathbb{H}^{n\times n}: \sum_{1\leqslant j_1\leqslant j_2<\cdots<j_k\leqslant n}\lambda_{j_1}(S)\,\lambda_{j_2}(S)\cdots\lambda_{j_k}(S)\leqslant 1\right\},$$
$$1\leqslant k\leqslant n.$$

For $k=1$, $\mathscr{H}_1^{(1)}$ is just the subset $\{S\in\mathbb{H}^{n\times n}:\sum_{j=1}^n\lambda_j(S)\leqslant 1\}$, while for $k=n$, $\mathscr{H}_1^{(n)}=\{S\in\mathbb{H}^{n\times n}:\det S\leqslant 1\}$. For these two cases we were able to prove that the sets are Chebyshevian; we determined the el. b. appr. for a matrix A in $\mathbb{H}^{n\times n}\backslash\mathscr{H}_1^{(n)}$ and $\mathbb{H}^{n\times n}\backslash\mathscr{H}_1^{(n)}$, respectively, and verified that the metric projection is globally Lipschitz-continuous. We will spare the reader with details.

REFERENCES

1. J. Anderson, On normal derivations, *Proc. Amer. Math. Soc.* **38** (1973), 135–140.
2. Ph. Benilan and H. Attouch, Opérateurs accrétifs, *in* "Sém. sur les semi-groups et les opérateurs non linéaires," Exposé no. 1., Publ. Math. d'Orsay, Paris, 1970–1971.
3. H. Berens and M. Finzel, A continuous selection of the metric projection in matrix spaces, *in* "Numerical Methods of Approximation Theory, Vol. 8" (L. Collatz, G. Meinardus, and G. Nürnberger, Eds.), ISNM, Vol. 81, pp. 21–29, 1987; A continuous selection of the metric projection in matrix spaces, Addendum, *Numer. Math.* **57** (1990), 663–667.
4. H. Berens and H. J. Schmid, An example of a Chebyshev set: The complex case, *Rocky Mountain J. Math.* **19** (1988), 67–71.
5. P. R. Halmos, Positive approximants of operators, *Indiana Math. J.* **21** (1972), 951–960.
6. R. A. Horn and Ch. R. Johnson, "Matrix Analysis," Cambridge University Press, Cambridge, 1985.
7. A. G. Robertson, Strongly Chebyshev subspaces of matrices, *J. Approx. Theory* **55** (1988), 264–269.